JOURNAL OF APPROXIMATION THEORY 41, 159–169 (1984)

On Strong Uniqueness in Linear Complex Chebyshev Approximation

HANS-PETER BLATT

Lehrstuhl für Mathematik-Angewandte Mathematik, Katholische Universität Eichstätt, Ostenstraße 18, D-8078 Eichstätt, West Germany

Communicated by T. J. Rivlin

Received December 3, 1982; revised August 10, 1983

Let Q be a compact subset of $\mathbb{C}$ and $C(Q)$ the set of all continuous functions f: $Q \to \mathbb{C}$. A given function $f \in C(Q)$ is approximated with respect to the uniform norm by elements of an n-dimensional Haar subspace V. Though the best approximation is in general not strongly unique, it can be shown that strong uniqueness is a generic property if and only if the compact set Q has at most n isolated points.

1. STRONG UNIQUENESS

Let Q be a compact subset of the complex plane $\mathbb{C}$, let $C(Q)$ denote the set of continuous complex valued functions endowed with the uniform norm $\|\cdot\|$, and let V be an n-dimensional Haar subspace of $C(Q)$:

$$V = \operatorname{span}\{v_1, v_2, \ldots, v_n\}. \tag{1.1}$$

We best approximate a given $f \in C(Q)$ with respect to V, i.e., we want to determine a $\tilde{v} \in V$ such that

$$\|f - \tilde{v}\| = \min_{v \in V} \|f - v\|.$$

Let $E_f(v)$ denote the set of extremal points of the error function $f - v$,

$$E_f(v) := \{z \in Q \mid \|f - v\| = |f(z) - v(z)|\}, \tag{1.2}$$

Let $\operatorname{Re} y$ ($\operatorname{Im} y$) be the real part (imaginary part) of $y \in \mathbb{C}^n$. We define for any $\varphi \in [-\pi, \pi]$ and $z \in Q$:

$$w(z, \varphi) = e^{i\varphi}(v_1(z), \ldots, v_n(z)) \in \mathbb{C}^n, \tag{1.3}$$

$$a(z, \varphi) = (\operatorname{Re} w(z, \varphi), -\operatorname{Im} w(z, \varphi)) \in \mathbb{R}^{2n}. \tag{1.4}$$

0021-9045/84 $3.00

Let us consider for each $v \in V$ the representation

$$v = \sum_{j=1}^{n} (\alpha_j + i\alpha_{j+n}) v_j. \tag{1.5}$$

Then we get

$$\operatorname{Re}(e^{i\varphi} v(z)) = \langle a(z, \varphi), \alpha \rangle \tag{1.6}$$

with $\alpha = (\alpha_1, ..., \alpha_{2n}) \in \mathbb{R}^{2n}$ and $\langle\ ,\ \rangle$ the usual inner product in $\mathbb{R}^{2n}$.

The vectors $a(z, \varphi)$ play an important role for characterizing best approximations. Instead of extremal signatures (Rivlin and Shapiro [12], Brosowski [2]) we use a generalization of the notion of "reference" of Stiefel [14].

DEFINITION 1. $R = \{(z_j, \varphi_j) \mid 1 \leqslant j \leqslant m\}$ is called a *reference* if there exist positive numbers $\lambda_1, \lambda_2, ..., \lambda_m$ such that

$$\sum_{j=1}^{m} \lambda_j = 1, \tag{1.7}$$

$$\operatorname{rank}(A) = m - 1, \tag{1.8}$$

$$\lambda \cdot A = 0, \tag{1.9}$$

with

$$A = \begin{pmatrix} a(z_1, \varphi_1) \\ \vdots \\ a(z_m, \varphi_m) \end{pmatrix} \quad \text{and} \quad \lambda = (\lambda_1, ..., \lambda_m). \tag{1.10}$$

The numbers λ_j $(1 \leqslant j \leqslant m)$ are called *characteristic numbers of the reference R*. Because of (1.7) and (1.8) these numbers are uniquely determined.

DEFINITION 2. A reference R is *a reference with respect to the error function* $f - v$, if

$$e^{i\varphi}(f - v)(z) = \|f - v\|,$$

for each $(z, \varphi) \in R$.

Then we get the following simple and concise characterization of the best approximation $\tilde{v}$.

THEOREM 1. *$\tilde{v}$ is a best approximation to f if and only if there exists a reference R with respect to $f - \tilde{v}$.*

It is well known that the best approximation is unique. Each reference R with respect to the optimal error function $f - \tilde{v}$ satisfies the inequalities:

$$n + 1 \leqslant |R| \leqslant 2n + 1.$$

From the numerical point of view it is important to know sharper results about the quality of uniqueness. It would be desirable if the best approximation could be *strongly unique.*

DEFINITION 3. The best approximation $\tilde{v}$ to f is *strongly unique* if there exists a real number $\gamma > 0$ such that

$$\|f - v\| \geqslant \|f - \tilde{v}\| + \gamma \|v - \tilde{v}\|$$

for each $v \in V$.

However, Newman and Shapiro [9] have shown that the best approximation $\tilde{v}$ to f is in general not strongly unique: For any bounded neighbourhood U of $\tilde{v}$ there exists a real number $\gamma > 0$ such that

$$\|f - v\| \geqslant \|f - \tilde{v}\| + \gamma \|v - \tilde{v}\|^2. \tag{1.11}$$

The inequality (1.11) is sharp. Strong uniqueness for the best approximation is closely related to the number of points of the references. One can show the following

THEOREM 2. *$\tilde{v}$ is a strongly unique best approximation to f if and only if there exist a finite number of references R_k $(k = 1,..., s)$ with respect to $f - \tilde{v}$ such that the vector space $\mathbb{R}^{2n}$ is the span of*

$$\left\{ a(z, \varphi) \mid (z, \varphi) \in \bigcup_{k=1}^{s} R_k \right\}.$$

(For a proof see Brosowski [3].) As a consequence we get a result of Gutknecht [6]: If there exists a reference R with respect to $f - \tilde{v}$ such that $|R| = 2n + 1$, then $\tilde{v}$ is strongly unique.

2. STRONG UNIQUENESS AS A GENERIC PROPERTY

We denote by

$$m(f) := \min\{|R| \mid R \text{ reference with respect to } f - \tilde{v}\}, \tag{2.1}$$

so $m(f)$ is the minimal length of a reference with respect to the optimal error function $f - \tilde{v}$. Defining

$$T_k := \{f \in C(Q) \mid m(f) = k\} \tag{2.2}$$

we can formulate our main result as

THEOREM 3. *T_{2n+1} is open and dense in $C(Q)$ if and only if Q has at most n isolated points.*

The main tool for proving Theorem 3 is the following lemma.

MAIN LEMMA. *If Q has at most n isolated points and if $f \in T_m$, $m \leqslant 2n$, then there exists for any $\varepsilon > 0$ a function $\tilde{f} \in T_{m+1}$ such that*

$$\|f - \tilde{f}\| \leqslant \varepsilon.$$

Proof of Theorem 3. Let Q have at most n isolated points. Then by the preceding lemma, T_{2n+1} is a dense subset of $C(Q)$. Let us assume that T_{2n+1} is not open; then there exists a function $f \in T_{2n+1}$ and a sequence $f_n \to f$ such that $f_n \notin T_{2n+1}$. Denoting by $\tilde{v}_n$ the best approximation of f_n we may find a reference R_n with respect to $f_n - \tilde{v}_n$ such that $|R_n| \leqslant 2n$. Because Q is compact we may assume without loss of generality that $R_n \to R$ as $n \to \infty$. Moreover the best approximation $\tilde{v}_n$ converges to the best approximation $\tilde{v}$ of f. Thus R or a subset of R is a reference with respect to $f - \tilde{v}$. As $|R| \leqslant 2n$ this contradicts $f \in T_{2n+1}$.

Conversely, let Q be a set with at least $n+1$ isolated points $z_1, z_2, \ldots, z_{n+1}$. We determine $\lambda_j \in \mathbb{C}$ $(1 \leqslant j \leqslant n+1)$ such that

$$\sum_{j=1}^{n+1} |\lambda_j| = 1 \qquad \text{and} \qquad \sum_{j=1}^{n+1} \lambda_j v(z_j) = 0$$

for each $v \in V$. Since V is a Haar subspace, each λ_j is different from zero. Using the extension theorem of Tietze [8] we get a function $f \in C(Q)$ such that $\|f\| = 1$, $|f(z)| < 1$ for each $z \neq z_j$, and

$$\frac{\lambda_j}{|\lambda_j|} f(z_j) = 1 \qquad \text{for} \quad j = 1, 2, \ldots, n+1.$$

From Theorem 1 one concludes that 0 is the best approximation of f and

$$E_f(0) = \{z_1, z_2, \ldots, z_{n+1}\}.$$

Since each z_j is an isolated point, there exists a neighbourhood U of f such that for the best approximation v_g of $g \in U$ we have

$$E_g(v_g) = \{z_1, z_2, ..., z_{n+1}\}.$$

Hence $U \subset T_{n+1}$ and therefore T_{2n+1} is not dense in $C(Q)$.

From Theorem 3 we may deduce that strong unicity holds almost everywhere in $C[Q]$. To be more precise we use the following topological terminology [10].

DEFINITION 4. A subset A of a topological space X is *residual* iff A is the intersection of a countable family of open and dense subsets of X. A property is called *generic* iff this property holds for a residual subset of X.

Hence we get from Theorem 3—keeping in mind the second part of the proof to this theorem—the following.

COROLLARY. *The property "f has a strongly unique best approximation" is generic in $C[Q]$ if and only if Q has at most n isolated points.*

We want to emphasize that the set T_{2n+1} is in general properly contained in the class of functions with strongly unique best approximations. For example, the function $f(z) = z^n$ has 0 as a strongly unique best approximation on the unit disk, but $f(z) \in T_{n+1}$ [13] (compare the final remarks in the last section).

3. Proof of the Main Lemma

Let $R = \{(z_j, \varphi_j) \mid 1 \leqslant j \leqslant m\}$ be a reference with respect to $f - \tilde{v}$ satisfying (1.7)–(1.9). We divide the proof into two steps:

(A) First we construct a reference R_1, $|R_1| = m + 1$.

$$R_1 = \{(z_j, \tilde{\varphi}_j) \mid 0 \leqslant j \leqslant m\},$$

such that

$$|(f - \tilde{v})(z_j) - e^{-i\tilde{\varphi}_j} \|f - v\|| \leqslant \frac{\varepsilon}{2} \qquad \text{for} \quad j = 0, ..., m. \tag{3.1}$$

(B) Then we show that there exists $\tilde{f} \in C(Q)$ such that $\|\tilde{f} - f\| \leqslant \varepsilon$, $E_{\tilde{f}}(\tilde{v}) = R_1$, and R_1 is a reference with respect to $\tilde{f} - \tilde{v}$.

In order to prove (A) we use the fact that at most n points of R are isolated. Therefore we may assume that the points $z_1, z_2, ..., z_{m-n}$ are not isolated. Defining for $(z, \varphi) \in Q \times [-\pi, \pi)$,

$$b(z, \varphi) = a\left(z, \varphi - \frac{\pi}{2}\right), \tag{3.2}$$

we consider the matrix

$$B_1 = \begin{bmatrix} a(z_1, \varphi_1) \\ \vdots \\ a(z_n, \varphi_n) \\ b(z_1, \varphi_1) \\ \vdots \\ b(z_{m-n}, \varphi_{m-n}) \end{bmatrix} \in R_m^{2n}.$$

Since V is a Haar subspace, we conclude that $\operatorname{rank}(B_1) = m$.

Hence it is possible to extend A (A defined in (1.10)) by a vector $b(z_k, \varphi_k)$, $1 \leqslant k \leqslant m-n$, to a matrix B_2 of rank m. We may assume that $k = 1$,

$$B_2 = \begin{pmatrix} a(z_1, \varphi_1) \\ b(z_1, \varphi_1) \\ a(z_2, \varphi_2) \\ \vdots \\ a(z_m, \varphi_m) \end{pmatrix} \in \mathbb{R}_{m+1}^{2n}. \tag{3.4}$$

With the same argument as above the matrix

$$B_3 = \begin{pmatrix} a(z_1, \varphi_1) \\ b(z_1, \varphi_1) \\ \vdots \\ a(z_n, \varphi_n) \\ b(z_n, \varphi_n) \end{pmatrix} \in \mathbb{R}_{2n}^{2n} \tag{3.5}$$

has $\operatorname{rank}(B_3) = 2n$. Hence, reordering the points z_j, $2 \leqslant j \leqslant n$, we may suppose that the matrix

$$B_4 = \begin{bmatrix} a(z_1, \varphi_1) \\ b(z_1, \varphi_1) \\ \vdots \\ a(z_{2n-m+1}, \varphi_{2n-m+1}) \\ b(z_{2n-m+1}, \varphi_{2n-m+1}) \\ a(z_{2n-m+2}, \varphi_{2n-m+2}) \\ \vdots \\ a(z_m, \varphi_m) \end{bmatrix} = \begin{pmatrix} a(z_1, \varphi_1) \\ b(z_1, \varphi_1) \\ C \end{pmatrix} \tag{3.6}$$

with $C \in \mathbb{R}_{2n-1}^{2n}$ satisfies $\operatorname{rank}(B_4) = 2n$.

Next, we define for $\varphi \in (0, \pi/2)$,

$$
\begin{aligned}
B_5 &= \begin{pmatrix} \cos\varphi\, a(z_1, \varphi_1) - \sin\varphi\, b(z_1, \varphi_1) \\ \cos\varphi\, a(z_1, \varphi_1) + \sin\varphi\, b(z_1, \varphi_1) \\ C \end{pmatrix} \\
&= \begin{pmatrix} a(z_1, \varphi_1 + \varphi) \\ a(z_1, \varphi_1 - \varphi) \\ C \end{pmatrix}.
\end{aligned} \tag{3.7}
$$

Then $\operatorname{rank}(B_5) = 2n$ and

$$\frac{\lambda_1}{2\cos\varphi}\{a(z_1, \varphi_1 + \varphi) + a(z_1, \varphi_1 - \varphi)\} + \sum_{j=2}^{m} \lambda_j a(z_j, \varphi_j) = 0. \tag{3.8}$$

Replacing in the first row of B_5 the point z_1 by a point $z_0 \in Q$ nearby z_1 we get

$$B_6 = \begin{pmatrix} a(z_0, \varphi_1 + \varphi) \\ a(z_1, \varphi_1 - \varphi) \\ C \end{pmatrix} \tag{3.9}$$

Using continuity arguments we may fix $\varphi > 0$ and a real number $\delta > 0$ such that for $|z_0 - z_1| < \delta$ the inequality

$$|(f - \tilde{v})(z_k) - e^{i(\varphi_1 + \psi_k)} \|f - \tilde{v}\|| \leqslant \frac{\varepsilon}{2} \tag{3.10}$$

holds for $k = 0, 1$ with $\psi_0 = -\psi_1 = \varphi$. Now let us assume that δ is small enough such that $\operatorname{rank}(B_6) = \operatorname{rank}(B_5) = 2n$ if $|z_0 - z_1| < \delta$, and, moreover, in view of (3.8) any nontrivial solution of the homogeneous equation

$$(\mu_0, \mu_1, ..., \mu_{2n}) \cdot B_6 = 0 \tag{3.11}$$

satisfies $\mu_0 \neq 0$, $\mu_1 \neq 0$, $\operatorname{sgn}(\mu_0) = \operatorname{sgn}(\mu_1)$. Considering a solution of (3.11) with $\mu_0 = \lambda_1/2\cos\varphi$ we may write the part with respect to the row vectors $a(z_j, \varphi_j)$ and $b(z_j, \varphi_j)$ as

$$
\begin{aligned}
0 &= (\mu_0, \mu_1, ..., \mu_{2n}) \cdot B_6 \\
&= \sum_{j=0}^{m} (\alpha_j a(z_j, \varphi_j) + \beta_j b(z_j, \varphi_j)) \\
&= \sum_{j=0}^{m} \kappa_j a(z_j, \varphi_j + \psi_j)
\end{aligned}
$$

with $\kappa_0 = \mu_0$, $\kappa_1 = \mu_1$, $\kappa_j > 0$, and $\psi_j \in [-\pi, \pi)$ for $2 \leqslant j \leqslant m$. Thus we get a positive solution of the equation

$$(\kappa_0, \kappa_1, ..., \kappa_m) \cdot B_7 = 0 \tag{3.12}$$

where

$$B_7 = \begin{pmatrix} a(z_0, \varphi_1 + \psi_0) \\ a(z_1, \varphi_1 + \psi_1) \\ \vdots \\ a(z_m, \varphi_m + \psi_m) \end{pmatrix}.$$

If $z_0 \to z_1$ we get $\alpha_j \to \lambda_j$ and $\beta_j \to 0$; hence $\psi_j \to 0$ for $2 \leqslant j \leqslant m$. Moreover, $\operatorname{rank}(B_7) \to \operatorname{rank}(B_2) = m$. So $\operatorname{rank}(B_7) = m$ if z_0 is close enough to z_1. Summarizing: For any $\varepsilon > 0$ we may find $z_0 \in Q$, real numbers ψ_j, $-\pi \leqslant \psi_j < \pi$, and positive numbers κ_j such that (3.12), $\operatorname{rank}(B_7) = m$ and

$$|(f - \tilde{v})(z_j) - e^{i(\varphi_j + \psi_j)} \|f - \tilde{v}\|| \leqslant \frac{\varepsilon}{2} \tag{3.13}$$

hold for $j = 0, ..., m$. Thus the set

$$R_1 = \{(z_j, \tilde{\varphi}_j) : 0 \leqslant j \leqslant m, \tilde{\varphi}_j = \varphi_j + \psi_j\}$$

is a reference. The step (A) is proved.

Concerning (B) there exists, using the extension theorem of Tietze [8], a function $g \in C(Q)$ such that

$$(g - \tilde{v})(z_j) = e^{i\tilde{\varphi}_j} \|f - \tilde{v}\| \qquad \text{for } j = 0, ..., m. \tag{3.14}$$

$$\|g - \tilde{v}\| = \|f - \tilde{v}\|. \tag{3.15}$$

$$|(g - \tilde{v})(z)| < \|f - \tilde{v}\| \qquad \text{for each } z \in Q - \{z_0, ..., z_m\}. \tag{3.16}$$

Now we get from (3.13) and (3.14),

$$|(f - g)(z_j)| \leqslant \frac{\varepsilon}{2}. \tag{3.17}$$

Next, we have to show that there exists such a $g \in C(Q)$ with $\|f - g\| < \varepsilon$. Let us, therefore, define

$$M = \{g \in C(Q) \mid g \text{ satisfies (3.14)–(3.16)}\}. \tag{3.18}$$

M is a nonvoid and convex set. We assert that

$$\inf_{g \in M} \|f - g\| \leqslant \frac{\varepsilon}{2}. \tag{3.19}$$

If $g \in M$ and $\|f-g\| = \varepsilon/2 + \delta$ with $\delta > 0$, we set

$$S := \left\{ z \in Q \mid |(f-g)(z)| \geqslant \frac{\varepsilon}{2} + \frac{\delta}{2} \right\}.$$

Hence by (3.17), $z_j \notin S$ for $j = 0,..., m$. Again using the theorem of Tietze [8], there exists a function $h \in C(Q)$ such that

$$\begin{aligned} h(z) &= f(z) && \text{for each } z \in S, \\ (h - \tilde{v})(z_j) &= e^{-i\tilde{\varphi}_j} \|f - \tilde{v}\| && \text{for } j = 0,..., m, \end{aligned}$$

and

$$\|h - \tilde{v}\| = \|f - \tilde{v}\|.$$

Defining for $\alpha \in (0, 1)$

$$g_\alpha = \alpha \cdot g + (1-\alpha) \cdot h, \tag{3.20}$$

it is readily seen that $g_\alpha \in M$ and for $z \in S$ we have

$$|(f - g_\alpha)(z)| = |\alpha(f-g)(z)| \leqslant \alpha \left(\frac{\varepsilon}{2} + \delta \right). \tag{3.21}$$

For $z \in Q - S$ we get

$$\begin{aligned} |(f - g_\alpha)(z)| &\leqslant |(f-g)(z)| + (1-\alpha)\, |(g-h)(z)| \\ &\leqslant \frac{\varepsilon}{2} + \frac{\delta}{2} + 2(1-\alpha)\, \|f - \tilde{v}\|. \end{aligned} \tag{3.22}$$

Fixing

$$\alpha = \max \left(\frac{1}{2}, 1 - \frac{\delta}{8} \frac{1}{\|f - \tilde{v}\|} \right),$$

we get for $z \in Q - S$

$$|(f - g_\alpha)(z)| \leqslant \frac{\varepsilon}{2} + \frac{3}{4} \delta.$$

For $z \in S$ we get either $|(f - g_\alpha)(z)| \leqslant \varepsilon/4 + \delta/2$ or

$$\begin{aligned} |(f - g_\alpha)(z)| &\leqslant \frac{\varepsilon}{2} + \delta - \frac{\delta}{8} \frac{1}{\|f - \tilde{v}\|} \left(\frac{\varepsilon}{2} + \delta \right) \\ &\leqslant \frac{\varepsilon}{2} + \delta \left(1 - \frac{\varepsilon}{16\, \|f - \tilde{v}\|} \right). \end{aligned}$$

Summarizing, we get

$$\|f - g_\alpha\| \leqslant \frac{\varepsilon}{2} + \kappa \cdot \delta, \tag{3.23}$$

where $\kappa = \max(\frac{3}{4}, 1 - \varepsilon/16\,\|f - \tilde{v}\|)$,

Since κ is independent of δ, the assertion (3.19) follows from (3.23). Hence there exists $\tilde{f} \in C(Q)$ such that (3.14)–(3.16) and $\|f - \tilde{f}\| < \varepsilon$ hold. Thus $\tilde{v}$ is the best approximation of $\tilde{f}$, $E_{\tilde{f}}(\tilde{v}) = R_1$, and R_1 is a reference with respect to $\tilde{f} - \tilde{v}$. Step (B) and the lemma are proved.

4. Remarks

A given error function is optimal if and only if there exists an associated reference. If f is analytic in a region Q and $V = \Pi_n$, the set of polynomials of degree $\leqslant n$, then many of the known best approximations are not strongly unique, since the error curve has less than $2n + 3$ extremal points (Rivlin and Weiss [13], Geiger and Opfer [5]). Ryzakow considers Zolotarev's problem on the unit disk: though the approximations given in [14] fail to be optimal in general, one may easily prove that the optimal error curve has exactly $n + 2$ extremal points (or Zolotarev's problem coincides with the Chebyshev problem). Hence for Zolotarev's problem the best approximations are not strongly unique.

On the other hand there exist problems on the unit disk such that the optimal error curve is a perfect circle with winding number $n + 1$ (Al'per [1], Klotz [7], Trefethen [15]). Using Theorem 2 it is readily seen that in this case the best approximation is strongly unique. But this phenomenon occurs very rarely since the function f has to be a rational function.

Therefore it is an open question whether Theorem 3 is true or not if one restricts the approximation to analytic functions f in a Jordan region, for example, the unit disk, with respect to polynomials.

References

1. S. Ja. Al'per, Asymptotic values of best approximation of analytic functions in a complex domain, *Uspehi Mat. Nauk* **14** (1959), 131–134.
2. B. Brosowski, "Nichtlineare Tschebysheff-Approximation," Bibliographisches Institut, Mannheim, 1968.
3. B. Brosowski, A refinement of the Kolmogorov-Criterion, *in* "Proceedings, Conference of Approximation Theory, Varna, June 1981," in press.
4. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.

5. C. GEIGER AND G. OPFER, Complex Chebyshev polynomials on circular sections. *J. Approx. Theory* **24** (1978), 93–118.
6. M. H. GUTKNECHT, Non-strong uniqueness in real and complex Chebyshev approximation, *J. Approx. Theory* **23** (1978), 204–213.
7. V. KLOTZ, "Polynomiale and rationale Tschebyscheff-Approximation in der komplexen Ebene," Dissertation, Erlangen, 1974.
8. K. KURATOWSKI, "Topology," Academic Press, New York, 1966.
9. D. J. NEWMAN AND H. S. SHAPIRO, Some theorems on Čebyšev approximation, *Duke Math. J.* **30** (1963), 673–681.
10. Z. NITECKI, "Differentiable Dynamics," MIT Press, Cambridge, Mass., 1971.
11. T. J. RIVLIN, Best uniform approximation by polynomials in the complex plane, "Approximation Theory III" (E. W. Cheney, Ed.), pp. 75–86, Academic Press, New York, 1980.
12. T. J. RIVLIN AND H. S. SHAPIRO, A unified approach to certain problems of approximation and minimization, *J. SIAM* **9** (1961), 670–699.
13. T. J. RIVLIN AND B. WEISS, Some best polynomial approximations, *Duke Math. J.* 35 (1968), 475–482.
14. I. JA. RYZAKOW, An analog for the disk of Zolotarev's problem, *Soviet Math. Dokl.* **15** (1974), 882–885.
15. L. N. TREFETHEN, Near circularity of the error curve in complex Chebyshev approximation, *J. Approx. Theory* **31** (1981), 344–367.